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A few remarks on blowing-up and connectedness

By *M. Brodmann* at Zürich

1. Introduction

In this note we present some results which are related to the connectedness of fibers obtained by blowing-up an algebraic variety. The paper has two parts, according to the two types of questions which shall be treated, namely: bounds for the connectedness-dimension of fibers and calculation of Stein-factors of certain blowing-up morphisms.

More precisely, in the first part we consider the blowing-up $X \xrightarrow{\pi_I} \operatorname{Spec}(R)$ of a noetherian local ring R with respect to an ideal I of R . We then choose a proper ideal J of R , which contains I and give bounds for the connectedness-dimension of the fiber $Y := \pi_I^{-1}(V(J))$. Here, for a closed subset Z of a noetherian scheme X , the *connectedness-dimension* $c(Z)$ of Z is defined by:

$$(1.1) \quad c(Z) := \min \{ \dim(W) \mid W \subseteq Z \text{ closed, } Z - W \text{ disconnected} \}.$$

Thereby, the dimension of the empty set \emptyset is defined as -1 and \emptyset is considered as disconnected. So it always holds $c(Z) \geq -1$ with equality iff Z is disconnected. Often, we also shall use the following description of $c(Z)$:

(1.2) $c(Z) = \min \{ \dim(Z_1 \cap Z_2) \}$, where Z_1 and Z_2 are both unions of irreducible components of Z and satisfy $Z = Z_1 \cup Z_2$.

The starting point to obtain our bounds is [5], (3.4), which gives a criterion for the connectedness and a bound on the connectedness-dimension for the fibers of an arbitrary quasihomogeneous morphism $\pi: X \rightarrow \operatorname{Spec}(R)$. We first deduce a bound which is formulated in terms of the formal extension of our blowing-up, (2.5). Then we prove the lemma (2.7) which — for any arbitrary noetherian scheme — gives a bound for the connectedness-dimension in terms of the depths of the local rings of the scheme. (Bounds similar to (2.7) have been given by Hartshorne [11], (2.3), (2.4).) Finally we get a bound for the connectedness-dimension of our fiber Y , which depends only on the depths of the localisations of R , provided that R is excellent, (2.9). We apply this to get very explicite bounds in case R is either normal or satisfies the Serre-property S_2 ,

(2. 10). Thereby the normal case may be considered as a sharpened version of Zariski's connectedness theorem [14]. Finally we get bounds for the connectedness-dimensions of tangent cones, (2. 11).

In the second part we consider *Macaulayfications* $\tilde{V} \xrightarrow{\tilde{\pi}} V$ of the type introduced by Faltings [6]. In particular we show that — even for surfaces — *there is in general no minimal proper birational Cohen-Macaulay (CM) model* (3. 10), (3. 11). More precisely we will see that if there is a non-trivial *finite* birational CM-model which is minimal among all such models, it is not minimal among all *proper* birational CM-models. This answers to the negative a corresponding question of [2]. It also shows that the Serre-property S_2 alone (contrary to normality) in general may not be realized by a minimal proper birational model. This answers to the negative a corresponding question of [1]. For simplicity we prove the needed triviality of Grothendieck-Stein-factorizations only for a special class of Macaulayfications. In fact this result holds for all Macaulayfications introduced in [2], [4] and [6].

We also want to point out that the hypothesis of excellence in (2. 9), (2. 10) (ii), (2. 11) (iii) may be replaced by the weaker condition that R is universally catenary and that its formal fibers have the Serre-property S_1 . Consequently, the estimates (2. 10) (ii) and (2. 11) (iii) hold for an arbitrary CM-ring resp. an arbitrary locally noetherian CM-scheme.

As for the unexplained terminology we refer to [10] and [12].

2. Blowing-up and tangent cones

Let (R, \mathfrak{m}) be a local noetherian ring. Let $I \subseteq R$ be an ideal. The *arithmetic rank* $r(I)$ of I is defined as the minimal number of elements of I which span an ideal whose radical is \sqrt{I} . Thus:

$$(2. 1) \quad r(I) = \min \{r \mid \exists a_1, \dots, a_r \in I \text{ with } \sqrt{(a_1, \dots, a_r)} = \sqrt{I}\}.$$

Let X be a noetherian scheme, $Z \subseteq X$ a closed subset and $x \in Z$ a point. Let $\mathcal{I}_x \subseteq \mathcal{O}_{X,x}$ be the vanishing ideal of Z at x . Then $r(\mathcal{I}_x)$ is called the *arithmetic rank of Z at x* , and denoted by $r_{X,x}(Z)$. Furthermore we put

$$(2. 2) \quad r_X(Z) := \max \{r_{X,x}(Z) \mid x \in Z\}.$$

Using this notation, we have the following result, which has been shown in [5].

(2. 3) Proposition. *Let $S = R \oplus R_1 \oplus \dots$ be a graded noetherian R -algebra and assume that $\hat{X} := \text{Proj}(\hat{R} \otimes_R S)$ is connected. Let $Z \subseteq \text{Spec}(R)$ be a closed set and let $\pi: X := \text{Proj}(S) \rightarrow \text{Spec}(R)$ be the canonical morphism. Then the fiber $Y := \pi^{-1}(Z)$ is connected and satisfies the inequality $c(Y) \geq c(\hat{X}) - r_X(Y) - 1$.*

$$(2.4) \quad Bl_R(I) := \text{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi_I} \text{Spec}(R).$$

As an application of (2.3) we obtain

(2. 5) Proposition. *Let $I \subseteq J \subseteq \mathfrak{m}$ be ideals. Assume that $\hat{X} := \text{Bl}_{\hat{R}}(I\hat{R})$ is connected. Then $Y := \pi_I^{-1}(V(J))$ is connected and satisfies the inequality $c(Y) \geq c(X) - r(J/I) - 2$.*

Proof. Applying (2.3) with $S=R(I)$ and $Z=V(I)$ it remains to show that $r_{X,x}(Y) \leq r(J/I) + 1$ for all $x \in Y$ (where $X = B|_R(I)$). So, put $r = r(J/I)$ and let $a_1, \dots, a_r \in J$ be such that $\sqrt{(a_1, \dots, a_r)/I} = \sqrt{J/I}$. Let $x \in Y$. Then $I\mathcal{O}_{X,x}$ is a principal ideal of $\mathcal{O}_{X,x}$, [7]. Choosing a generator $b \in \mathcal{O}_{X,x}$ of $I\mathcal{O}_{X,x}$ we obtain $\sqrt{J\mathcal{O}_{X,x}} = \sqrt{(a_1, \dots, a_r, b)\mathcal{O}_{X,x}}$. This proves our claim as $\sqrt{J\mathcal{O}_{X,x}}$ is the vanishing ideal of Y at x .

We now want to give a connectedness-result for the fiber Y in which the formal extension $Bl_{\hat{R}}(I\hat{R})$ of $Bl_R(I)$ does not occur. First we will prove some auxiliary results.

For a noetherian scheme X we put

$$(2.6) \quad \delta(X) := \min \{ \dim \overline{\{x\}} \mid \text{depth}(\mathcal{O}_{X,x}) \leq 1 \}.$$

(2.7) Lemma. X connected $\Rightarrow c(X) \geq \delta(X)$.

Proof. Let $X_1, X_2 \subseteq X$ both be unions of irreducible components of X such that $X_1 \cup X_2 = X$. We have to show $d := \dim(X_1 \cap X_2) \geq \delta(X)$. Let x be a generic point of $X_1 \cap X_2$ such that $\dim \overline{\{x\}} = d$. We must show $\text{depth}(R) \leq 1$, where $R = \mathcal{O}_{X,x}$. We may assume that $\text{depth}(R) > 0$. Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be an irredundant primary decomposition of 0 in R . After an appropriate reordering of the primary components \mathfrak{q}_i we find a natural number $s < n$ such that $\sqrt{\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s}$ and $\sqrt{\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n}$ are the vanishing ideals of X_1 resp. of X_2 at x . Put $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, $J = \mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n$. It follows $I \cap J = (0)$, $\sqrt{I+J} = \mathfrak{m}$, $\text{depth}(R/I) > 0$, $\text{depth}(R/J) > 0$. Applying local cohomology to the exact sequence $0 \rightarrow R \rightarrow R/I \oplus R/J \rightarrow R/(I+J) \rightarrow 0$ we obtain an exact sequence

$$H_m^0(R/I) \oplus H_m^0(R/J) \rightarrow H_m^0(R/(I+J)) \rightarrow H_m^1(R).$$

As $\text{depth}(R/I), \text{depth}(R/J) > 0$ here the left hand term vanishes. As $\sqrt{I+J} = m$ the middle term equals $R/(I+J) \neq 0$. It follows $H_m^1(R) \neq 0$, thus $\text{depth}(R) \leq 1$.

In the sequel we use the notations

$c(R) := c(\text{Spec}(R))$, $\delta(R) := \delta(\text{Spec}(R))$. Zentralbibliothek Zürich

(2. 8) Lemma. Let $I \subseteq \mathfrak{m}$ be an ideal such that $ht(I) > 0$ and such that $\dim(R/I) < c(R)$. Then $c(Bl_R(I)) \geq c(R)$.

Proof. Put $X = Bl_R(I)$ and let $X = X_1 \cup X_2$, where X_1 and X_2 are unions of irreducible components of X . We must show $\dim(X_1 \cap X_2) \geq c(R)$. As $ht(I) > 0$, $\pi: X \rightarrow \text{Spec}(R)$ is birational (and proper). So we have $\text{Spec}(R) = \pi(X_1) \cup \pi(X_2)$ and both $\pi(X_1)$ and $\pi(X_2)$ are unions of irreducible components of $\text{Spec}(R)$. So there is a point $x \in \pi(X_1) \cap \pi(X_2)$ such that $\dim \overline{\{x\}} \geq c(R)$. As $\dim(R/I) < c(R)$ we have $x \notin V(I)$. So $\pi^{-1}(\{x\})$ consists of a single point y , and it holds $\dim \overline{\{y\}} \geq \dim \overline{\{x\}}$. In particular we have $y \in X_1 \cap X_2$, which proves our claim.

(2. 9) Proposition. Let R be excellent and let $I \subseteq J \subseteq \mathfrak{m}$ be ideals such that $\dim(R/I) < \delta(R)$. Then $Y := \pi_I^{-1}(V(J))$ is connected and satisfies $c(Y) \geq \delta(R) - r(J/I) - 2$.

Proof. In view of (2. 5) we have to show that $c(Bl_R(I\hat{R})) \geq \delta(R)$. In view of (2. 7) and (2. 8) it remains to show that $ht(I\hat{R}) > 0$ and that $\delta(\hat{R}) \geq \delta(R)$. As $\dim(R/I) < \delta(R)$ we have $ht(I) > 0$, thus $ht(I\hat{R}) > 0$. It remains to prove the second inequality. So, let $\hat{p} \in \text{Spec}(\hat{R})$ such that $\dim(\hat{R}/\hat{p}) = \delta(\hat{R})$ and $\text{depth}(\hat{R}_{\hat{p}}) \leq 1$. Put $\mathfrak{p} = \hat{p} \cap R$. As R is excellent, one of the two following statements holds [8], [12]:

$$\text{a) } \text{depth}(R_{\mathfrak{p}}) = \text{depth}(\hat{R}_{\hat{p}}), \dim(R/\mathfrak{p}) = \dim(\hat{R}/\hat{p}).$$

$$\text{b) } \text{depth}(R_{\mathfrak{p}}) = 0, \dim(R/\mathfrak{p}) = \dim(\hat{R}/\hat{p}) + 1.$$

a) immediately induces $\delta(\hat{R}) \geq \delta(R)$. If b) holds, we have $\dim(\hat{R}/\hat{p}) > 0$, thus $\dim(R/\mathfrak{p}) > 1$. We thus find a $\mathfrak{q} \in V(\mathfrak{p}) - \text{Ass}(R)$ such that $ht(\mathfrak{q}/\mathfrak{p}) = 1$. It follows $\text{depth}(R_{\mathfrak{q}}) = 1$, $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1 = \delta(\hat{R})$, thus $\delta(\hat{R}) \geq \delta(R)$.

(2. 10) Corollary. Let R be excellent, let $I \subset J \subset \mathfrak{m}$ be ideals and put $Y = \pi_I^{-1}(V(J))$.

(i) If R is normal and $I \neq 0$, then Y is connected and satisfies $c(Y) \geq \dim(R) - r(J/I) - 2$.

(ii) If R satisfies the Serre-property S_2 and if $ht(I) > 1$, then Y is connected and satisfies $c(Y) \geq \dim(R) - r(J/I) - 3$.

Proof. (i) If R is normal, \hat{R} is a domain [12] and so (as $I \neq 0$) $Bl_{\hat{R}}(I\hat{R})$ is integral and of the same dimension as R . Now apply (2. 5).

(ii) As R satisfies S_2 it follows $\delta(R) \geq \dim(R) - 1$. Now we may conclude by (2. 9).

We denote by $CT_x(X)$ the tangent cone at a point x of a noetherian scheme X .

(2. 11) Corollary. Let X be excellent, $x \in X$. Then

$$\text{(i) } c(CT_x(X)) \geq \delta(R) - 2.$$

$$\text{(ii) } X \text{ normal at } x \Rightarrow c(CT_x(X)) \geq \dim(\mathcal{O}_{X,x}) - 1.$$

$$\text{(iii) } X \text{ satisfies } S_2 \text{ at } x \Rightarrow c(CT_x(X)) \geq \dim(\mathcal{O}_{X,x}) - 2.$$

Proof. Put $\mathcal{O}_{X,x} = R$. Then

$$CT_x(X) = \text{Spec}(\mathfrak{R}(\mathfrak{m})/(\mathfrak{m})) \text{ and } Y := \pi_m^{-1}(\text{Spec}(R/\mathfrak{m})) = \text{Proj}(\mathfrak{R}(\mathfrak{m})/(\mathfrak{m}))$$

show that $c(Y) = c(CT_x(X)) - 1$ and allow to prove our statements applying (2. 9) and (2. 10) in the special case where $I = J = \mathfrak{m}$.

3. Stein-factors and Macaulayfication

As in the previous section, let (R, \mathfrak{m}) be a local noetherian ring and let $S = R \oplus S_1 \oplus \dots$ be a graded noetherian R -algebra. We put $X = \text{Proj}(S)$. Then $\Gamma(\mathcal{O}_X)$ is a finite R -algebra and defines the Grothendieck-Stein-factor of the canonical morphism $\pi: X \rightarrow \text{Spec}(R)$, [13], [7].

So, there is a commutative diagram

$$(3.1) \quad \begin{array}{ccc} X := \text{Proj}(S) & \xrightarrow{\pi} & \text{Spec}(R) \\ & \searrow \pi_0 \quad \nearrow \gamma & \\ & X_0 = \text{Spec}(\Gamma(\mathcal{O}_X)) & \end{array}$$

such that π_0 is proper, γ is finite and the fiber $\pi_0^{-1}(\{y\})$ is connected for each $y \in \pi(X)$. We want to determine the Stein-factorization (3. 1) for certain classes of blowing-up.

First we recall an algebraic description of $\Gamma(\mathcal{O}_X)$. We use the functor $D_J := \varinjlim_n \text{Hom}(J^n, \cdot)$ of J -transform, J being an ideal of a noetherian ring A [8], [1]. Obviously $D_J(A)$ is an A -algebra. Now, putting $S_+ = S_1 \oplus S_2 \oplus \dots$ the S -algebra $D_{S_+}(S)$ is canonically graded (as an S -module) and it holds [7], [13]:

$$(3.2) \quad \Gamma(\mathcal{O}_X) = D_{S_+}(S)_0.$$

Thereby — for a graded S -module M — M_n denotes the R -module of elements of degree n .

From now on, let $I \subseteq R$ be an ideal. For an R -module M , $\Gamma_I(M)$ shall denote the I -torsion of M . Recall that $\Gamma_I(R) \subseteq R = \Gamma(\mathcal{O}_{\text{Spec}(R)})$ is the ideal of sections having support in $V(I)$ and that $D_I(R)$ is canonically isomorphic to $\Gamma(\mathcal{O}_{\text{Spec}(R) - V(I)})$ [8], [1]. Moreover, there is a natural exact sequence [9], [1]:

$$(3.3) \quad 0 \rightarrow \Gamma_I(M) \rightarrow M \rightarrow D_I(M) \rightarrow H^1_I(M) \rightarrow 0$$

We recall another property of I -transforms, namely (cf. [8], [1]):

(3.4) Let $x \in I$ be a non-zero-divisor with respect to $\bar{R} = R/\Gamma_I(R)$. Then

$$D_I(R) = \bigcup_{n \geq 0} (\bar{R} : I^n)_{\bar{R}_x}.$$

(3.5) **Lemma.** Let $ht(I) > 0$ and put $\bar{R} = R/\Gamma_I(R)$, $D = D_I(R)$, $X = Bl_R(I)$. Then

$$R \subseteq \bigcup_{n \geq 0} (I^n \bar{R} : I^n)_D = \Gamma(\mathcal{O}_X) \subseteq D = \Gamma(\mathcal{O}_{\text{Spec}(R) - V(I)}).$$

Proof. By (3.3) it holds $\bar{R} \subseteq D$. As $ht(I) > 0$ there is a $c \in I$ which is non-zero divisor with respect to \bar{R} . By (3.4) it follows $D = \bigcup_{n \geq 0} (\bar{R} : I^n)_{\bar{R}_c}$. So it remains to verify the equality $\bigcup_{n \geq 0} (I^n \bar{R} : I^n)_{\bar{R}_c} = \Gamma(\mathcal{O}_X)$. Let $c^* \in \mathfrak{R}(I) = \bigoplus_{n \geq 0} I^n =: S$ be the element $c \in I$ considered as homogeneous of degree one. Thus $c^* = (0, c, 0, \dots) \in S$. Then by our choice of c , c^* belongs to S_+ and is a non-zero divisor with respect to $S := S/\Gamma_{S_+}(S)$. So, by (3.2) and (3.4) we obtain $\Gamma(\mathcal{O}_X) = \left[\bigcup_{n \geq 0} (\bar{S} : (S_+)^n)_{(S_+)_c^*} \right]_0$. Observing that $\bar{S} = \mathfrak{R}(I\bar{R})$ it follows

$$\Gamma(\mathcal{O}_X) = \bigcup_{n \geq 0} ((I\bar{R})^n : (I\bar{R})^n)_{R_c},$$

thus our claim.

Now, let V be an irreducible variety of dimension $d > 1$ and let $p \in V$ be a closed point. Assume that $V - \{p\}$ consists only of Cohen-Macaulay points. We put $R = \mathcal{O}_{V,p}$, $\mathfrak{m} = \mathfrak{m}_{V,p}$. Then, according to Grothendieck's finiteness-theorem [9] the local cohomology-modules $H_{\mathfrak{m}}^i(R)$ are finitely generated for $i = 0, 1, \dots, d-1$. As a consequence there is an \mathfrak{m} -primary ideal \mathfrak{q} of R such that for each system of parameters $x_1, \dots, x_d \in \mathfrak{q}$ it holds [6], [2]:

$$(3.6) \quad (x_1, \dots, x_d) H_{\mathfrak{m}}^i(R/(x_1, \dots, x_j)) = 0, \quad \text{whenever } i + j < d.$$

We fix such a system $x_1, \dots, x_d \in \mathfrak{q}$ and put $I = (x_1, \dots, x_d)$. Then the blowing-up $\pi_I: Bl_R(I) \rightarrow \text{Spec}(R)$ is a *Macaulayfication*, e.g. $Bl_R(I)$ is a CM-scheme [6], [2]. Now let $\mathcal{J} \subseteq \mathcal{O}_V$ be the ideal defined by

$$\mathcal{J}_x = \begin{cases} \mathcal{O}_{V,x}, & \text{for } x \neq p, \\ I, & \text{for } x = p \end{cases}$$

and put $\tilde{V} := \text{Proj}(\mathfrak{R}(\mathcal{J}) = \bigoplus_{n \geq 0} \mathcal{J}^n)$. Then clearly the blowing-up $\tilde{\pi}: \tilde{V} \rightarrow V$ is a *Macaulayfication*.

(3.7) Proposition. *The Grothendieck-Stein-factorization of the Macaulayfication $\tilde{\pi}: \tilde{V} \rightarrow V$ is given by*

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\pi}} & V \\ & \searrow \tilde{\pi} & \nearrow \text{id} \\ & V & \end{array}$$

Proof. As $\tilde{\pi}: \tilde{V} - \tilde{\pi}^{-1}(\{p\}) \rightarrow V - \{p\}$ is an isomorphism, it suffices to show that the Grothendieck-Stein-factor of $\pi_I: Bl_R(I) \rightarrow \text{Spec}(R)$ is given by $\gamma = \text{id}: \text{Spec}(R) \rightarrow \text{Spec}(R)$. As V is integral we have $\Gamma_I(R) = 0$. So, as $D := D_m(R) = D_I(R)$ it remains to show that $\bigcup_{n \geq 0} (I^n: I^n)_D = R$, (3.5) (observe that $R \subseteq D$ by (3.4)). So we have to prove $(I^n: I^n)_D \subseteq R$ for all $n \geq 0$. To do so, put $L = (x_1, \dots, x_{d-1})R$. Then we know, that $(L^m: x_d)_D \subseteq L^{m-1}$ for all $m \in \mathbb{N}$, [2]. It follows

$$(I^m: x_d)_D = ((L^m + x_d I^{m-1}): x_d)_D = (L^m: x_d)_D + I^{m-1} = I^{m-1},$$

thus $(I^m: I)_D = I^{m-1}$ for all $m \in \mathbb{N}$. By induction on n we get immediately $(I^n: I^n)_D = R$ for all $n \geq 0$. This proves our claim.

In view of (3.3) it is clear that $D = D_m(R)$ is a finite birational integral extension of R and that

$$(3.8) \quad H_m^i(D) = \begin{cases} 0, & i \leq 1 \\ H_m^i(R), & i > 1 \end{cases} \quad (\text{cf. [3]}).$$

So, putting $V' = \text{Spec}(\mathcal{O}_{V-(p)})$, the canonical map $v: V' \rightarrow V$ is finite, birational and V' is of depth > 1 in all closed points.

(3.9) Corollary. *Let $\text{depth}(\mathcal{O}_{V,p}) = 1$. Then the Macaulayfication $\tilde{\pi}: \tilde{V} \rightarrow V$ does not factor through the finite morphism $v: V' \rightarrow V$.*

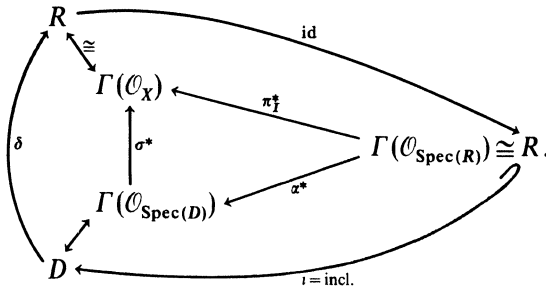
Proof. A factorization

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\pi}} & V \\ \downarrow & \searrow & \nearrow v \\ V' & & \end{array}$$

would induce a commutative diagram

$$(*) \quad \begin{array}{ccc} X := Bl_R(I) & \xrightarrow{\pi_I} & \text{Spec}(R). \\ \sigma \downarrow & \searrow \alpha(\text{can.}) & \\ \text{Spec}(D) & & \end{array}$$

From the previous proof we have the canonical isomorphism $\Gamma(\mathcal{O}_X) \cong R$. So, passing to global sections in (*) gives the commutative diagram



By (3.3) we have $\text{coker } \iota = H_m^1(R)$ and thus get a splitting exact sequence of R -modules

$$0 \rightarrow R \xrightarrow{\iota} D \rightarrow H_m^1(R) \rightarrow 0.$$

As D is a birational extension of R , ι must be an isomorphism. It follows $H_m^1(R) = 0$, thus $\text{depth}(\mathcal{O}_{V,p}) > 1$.

Keep the above hypotheses and notations. Then $D = D_m(R)$ is the least R -module of depth > 1 , containing R and contained in the quotient field of R [1], [2]. This shows that $V' \xrightarrow{\nu} V$ is the (essentially unique) minimal finite birational model of V whose closed points are of depth > 1 . In the special case, where $H_m^i(R) = 0$ for $i \neq 1, d$, (3.8) shows that D is the least CM- R -module containing R and being contained in the quotient field of R . So, in this situation, $V' \xrightarrow{\nu} V$ is the minimal *finite* birational model of V , which is CM (thus the minimal *finite* Macaulayfication). Observing this we get

(3.10) Corollary. *Let $\text{depth}(\mathcal{O}_{V,p}) = 1$. Then*

(i) *There is no minimal proper birational model $V^* \xrightarrow{\mu} V$, whose closed points are of depth > 1 .*

(ii) *If $H_{m_{V,p}}^i(\mathcal{O}_{V,p}) = 0$ for all $i \neq 1, d$, there is no minimal proper birational model $V^* \xrightarrow{\mu} V$ such that V^* is CM.*

Proof. Indeed, such a minimal model $V^* \xrightarrow{\mu} V$ would be a factor of the corresponding minimal finite birational model $V' \xrightarrow{\nu} V$, thus coincide with this latter. But then the Macaulayfication $\tilde{V} \xrightarrow{\tilde{\mu}} V$ would factor through $V' \xrightarrow{\nu} V$. This contradicts (3.9).

(3.11) Examples. (i) Let z and w be indeterminates and let

$$V = \text{Spec}(C[z, w^2, zw, w^3])$$

Let $p \in V$ be the closed point $z = w = 0$. It is immediate that

$$D_{m_{V,p}}(\mathcal{O}_{V,p}) = \mathbb{C}[z, w]_{(z,w)} = \mathcal{O}_{\mathbb{A}^2, o}$$

(where o denotes the origin of \mathbb{A}^2). So the normalization morphism $v: \mathbb{A}^2 \rightarrow V$ (which is induced by the inclusion $\mathbb{C}[z, w^2, zw, w^3] \hookrightarrow \mathbb{C}[z, w]$) is the minimal finite Macaulayfication of V . It is immediate to check that $H_m^1(\mathcal{O}_{V,p}) \cong \mathbb{C}$. This allows to choose the previously introduced ideal \mathfrak{q} as $\mathfrak{m} = \mathfrak{m}_{V,p}$, [3]. Putting

$$J = (z, w^2) \mathbb{C}[z, w^2, zw, w^3],$$

$\tilde{V} := \text{Proj}(\bigoplus_{n \geq 0} J^n) \xrightarrow{\tilde{\pi}} V$ is a Macaulayfication of V which does not factor through $v: \mathbb{A}^2 \rightarrow V$. It is easy to see that v is a homeomorphism. So, *topologically* $\tilde{\pi}$ factors through v .

(ii) Let $V = \text{Spec}(\mathbb{C}[z, zw, w^2 - w, w^3 - w])$ and let p be the point $z = w = 0$. It is easy to check that the normalization morphism $v: \mathbb{A}^2 \rightarrow V$ (induced by the inclusion $A := \mathbb{C}[z, zw, w^2 - w, w^3 - w] \hookrightarrow \mathbb{C}[z, w]$) is the minimal finite Macaulayfication of V , and that $H_{m_{V,p}}^1(\mathcal{O}_{V,p}) \cong \mathbb{C}$. So, putting $J = (z, w^2 - w)A$, $\tilde{\pi}: \tilde{V} = \text{Proj}(\bigoplus_{n \geq 0} J^n) \rightarrow V$ is a Macaulayfication which does not factor through $v: \mathbb{A}^2 \rightarrow V$. Here, *there is no surjective topological factorization*, as the fiber $v^{-1}(\{p\})$ consists of the two points $(0, 0), (1, 0) \in \mathbb{A}^2$, whereas the fiber $\tilde{\pi}^{-1}(\{p\})$ is homeomorphic to \mathbb{P}^1 , thus connected. Note, that $\text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}_{V,p}^n) \rightarrow V$ defines a Macaulayfication which factors through v by a proper map and whose exceptional fiber consists of two copies of \mathbb{P}^1 .

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